

Calculus III
Answers to and comments on the 2006 test
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Referring to the test version as on the course web page, the answers were

1	2
2	1 (2 marks for 3: see below)
3	3
4	3
5	2
6	2
7	3
8	scrapped, see below
9	1
10	2

Comments and details

Very unfortunately, this test contained both a mathematical error and a technical production error, affecting different questions. These, and the remedies adopted, are discussed below.

The errors may have contributed to the feeling of those people who have commented that they felt it was a bit harder than last year's. There are two other possible reasons. One is that the 'distractors' (the wrong answers intended to distract you from the right answer) may have been more distracting. The other is that a couple of the questions required a bit more calculation, mainly Q8 (which is the question which had to be discounted), though in overall time required for the test these should have been compensated by the questions requiring no calculation or only trivial calculation like Q3 and Q7.

Here are some remarks on individual questions. In the performance analysis answer 0 means no answer, 1 means the first choice, and so on. 98 scripts were received and the overall average score (out of 31) was 16.45 which translates to approximately 53%. This is a little low and I may ask the external to allow me to scale it up a little if the exam average comes out better. Note that 3 questions were got right by over 2/3 of candidates, and the questions on grad, div and curl were generally well done which is encouraging, while the Fourier series ones were conspicuously poorly done.

1. Testing knowledge of the behaviour of Fourier series at a discontinuity of the original function. At 0, $f(x) \rightarrow 4\pi^2$ if we come from below, and $\rightarrow 0$ if we come from above 0. (These values are the distractors.) The series takes the mean of the two, i.e. $2\pi^2$.

Performance: 0:27, 1:20, 2:25, 3:26. Surprisingly often avoided, and surprisingly badly done, I felt.

2. This question unfortunately contained a mathematical error due to me not checking my source carefully enough (the example was adapted from one in a book). The intended answer, 1, does not in general satisfy $y(\pi/2) = 1$: it has this value only if $k = 4n + 1$ for some integer n . A student who just checked the boundary values would find only answer 3 fitted, although they would not have done the intended calculus of variations calculation: I decided this deserved 2. Nobody commented that as set there is no correct answer. To get the question right I should have said $k \neq 0$, $y(\pi/2k) = 1$ and changed the third answer to $y = \sinh kx / \sinh(\pi/2)$.

Now for the intended answer...

Using the Euler-Lagrange equations gives $2(y')' + 2k^2y = 0$ [alternatively, using the Hamiltonian

gives $2y'^2 - (y'^2 - k^2y^2) = y'^2 + k^2y^2 = \text{constant}$]. Of the given answers, this is only solved by the trig function cases and answer 2 does not fit $y(0) = 0$.

Performance: 0:21, 1:39, 2:6, 3:32. Very few got no marks!

3. Not all points are possible, so not 1. The $\mathbf{k}$ component varies with t and need not be 0, so not 2. So even if you did not recognize the word 'helix' the answer is obviously 3.

Performance: 0:8, 1:16, 2:5, 3:69. The one with the highest number of correct answers.

4. Along the curve $d\mathbf{r} = (-2\sin t\mathbf{i} + \cos t\mathbf{j})dt$ and $\mathbf{F} = 6\cos t\mathbf{j} - 4\sin t\mathbf{k}$ so $\int \mathbf{F} \cdot d\mathbf{r} = \int 6\cos^2 t dt$ between $t = 0$ and $t = 2\pi$ so the answer is $\frac{1}{2}6(2\pi) = 6\pi$ (using a result for $\int \cos^2 t dt$ from the Fourier series chapter). So answer 3 is correct

Performance: 0:14, 1:5, 2:51, 3:28. A surprising number fell for the wrong answer 2.

5. Divergence must be a scalar so 3 is ruled out. In fact

$$\nabla \cdot \mathbf{F} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(3x - yz)}{\partial y} + \frac{\partial z^3}{\partial z}$$

The remaining distractor arises from wrongly taking

$$\nabla \cdot \mathbf{F} = \frac{\partial(x^2y + 3x - yz + 3z^3)}{\partial x} + \frac{\partial(x^2y + 3x - yz + 3z^3)}{\partial y} + \frac{\partial(x^2y + 3x - yz + 3z^3)}{\partial z}.$$

Performance: 0:2, 1:19, 2:53, 3:24. Pretty well done, overall, though about 1/4 of the answers gave a vector where there should be a scalar, and would otherwise have been correct.

6. Although $\sin x$ is odd, the half-range cosine series for the function equal to $\sin x$ on $(0, \pi)$ is actually the series for the even function $|\sin x|$. To get the coefficients we need to do $a_0 = \frac{2}{\pi} \int_0^\pi \sin x dx = \frac{2}{\pi} [-\cos x]_0^\pi = \frac{4}{\pi}$ and $a_1 = \frac{2}{\pi} \int_0^\pi \sin x \cos x dx = \frac{1}{\pi} \int_0^\pi \sin 2x dx = \frac{1}{2\pi} [-\cos 2x]_0^\pi = 0$.

Performance: 0:35, 1:33, 2:6, 3:17, 4:7. By far the worst done question. Too many people assumed that $\sin x$ on $[0, \pi]$ had to give an odd function, it seems.

7. The Lagrangian depends on all three of x , y and y' . The Hamiltonian is in general constant only if x is missing, so not 1 or 2. The Euler-Lagrange equation always applies, so not 4.

Performance: 0:6, 1:11, 2:8, 3:67, 4:6. The third highest fraction of correct answers.

8. This question was completely messed up by a production error which led to the program that creates randomized test papers picking up the first mistyped draft rather than the corrected question. This made 3 of the 4 choices correct. I decided the best way to remedy this was to ignore the question altogether and mark the paper out of 31 total.

To explain, define $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{n} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{v}_1 = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v}_2 = 5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. This last was misprinted and should have been $\mathbf{v}_2 = 5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Then the intended answer is as follows.

Choice 1. $\mathbf{n}$ has a zero dot product with each of $\mathbf{v}_1$ and corrected $\mathbf{v}_2$ so this is true

Choice 2. This is $\mathbf{r} = \mathbf{a} + t\mathbf{v}_1$ which must be in the plane.

Choice 3. Assuming the correct $\mathbf{v}_2$ so $\mathbf{n}$ is correct, this was meant to be $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, which gives $x - y + 2z = 1 - 1 - 2 = -2$, but was also misprinted.

Choice 4. The idea was that as the others are true, this must be the right answer. Actually, the unit normal should have been $\mathbf{n}/|\mathbf{n}| = \mathbf{n}/\sqrt{6}$ and the distance is then $|\mathbf{a} \cdot \mathbf{n}/\sqrt{6}| = 2/\sqrt{6}$. The distractor here is inviting you to take the length of $\mathbf{a}$.

Performance: 0:42, 1:14, 2:11, 3:23, 4:8. Disregarded, and many rightly avoided it, but I am a bit concerned by the 11 who picked the one incorrect answer that was not made correct by the errors in the question!

9. Gradient must be a vector, so not 3 or 4 (choice 3 was picked as a distractor because it is the

sum of the three correct components).

$$\begin{aligned}\nabla V &= \frac{\partial(x^2z^2 + y^3z)}{\partial x}\mathbf{i} + \frac{\partial(x^2z^2 + y^3z)}{\partial y}\mathbf{j} + \frac{\partial(x^2z^2 + y^3z)}{\partial z}\mathbf{k} \\ &= 2xz^2\mathbf{i} + 3y^2z\mathbf{j} + (2x^2z + y^3)\mathbf{k} \\ &= 2.1.2^2\mathbf{i} + 3.(-1)^2.2\mathbf{j} + (2.1.2 + (-1)^3)\mathbf{k}\end{aligned}$$

The remaining distractor, 2, is obtained by not being careful with the minuses arising on substituting the value of y .

Performance: 0:2, 1:68, 2:0, 3:21, 4:7. Second best done and a lot more people did the differentiation and substitution correctly but gave a scalar instead of a vector.

10. If you write down the determinant form you can see that the x^2 cannot be differentiated with respect to x and similarly for the y^2 and z^2 . The remaining bits leave

$$\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\mathbf{k}.$$

The first choice arises from differentiating each component by the associated variable, the 3rd from mistakenly differentiating the x^2 etc, and the last from a sign error in the otherwise correct version.

Performance: 0:7, 1:6, 2:55, 3:3, 4:27. A lot of the errors thus must have been sign errors by people who otherwise knew what they were doing.